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ON IRREDUCIBLE MODULES OVER q-SKEW POLYNOMIAL RINGS AND SMASH PRODUCTS

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Abstract. Let M be an irreducible left module over a q-skew polynomial ring $R[x;\sigma,\delta]$. We give sufficient conditions for the complete reducibility of M considered as a module over the coefficient ring R. We apply it to irreducible modules over smash product $R\#H$, where H is a Hopf algebra generated by skew primitive elements.

1. Introduction

For a given extension $R \subseteq S$ of associative rings (with the same unity), it is natural to ask whether (or when) irreducible left S-modules are completely reducible as R-modules. This question has a positive answer for several classes of "finite type" extensions: for example,

(i) finite normalizing extensions $R \subseteq \sum_{i=1}^{n} Rs_i$ ([2]),
(ii) fixed rings of a finite group actions $R^G \subseteq R$, with $|G|^{-1} \in R$ ([8]),
(iii) rings graded by finite groups $R_1 \subseteq \bigoplus_{g\in G} R_g$ ([4]).

In this paper we study some extensions of "infinite type"; namely, we consider modules over q-skew polynomial rings. We show that, under certain conditions, for a given left $R[x;\sigma,\delta]$-module M its socle $\mathrm{Soc}({}_RM)$ over R is also a module over the ring $R[x;\sigma,\delta]$. Our conditions imply in particular that if q is not a root of 1, then:

1. finite dimensional irreducible $R[x;\sigma,\delta]$-modules are completely reducible over R;
2. if R is left socular (e.g., left artinian or right perfect), then irreducible left $R[x;\sigma,\delta]$-modules are completely reducible over R.

As a consequence of our results on modules over q-skew polynomial rings, we obtain a description of certain modules over smash products $R\#H$, where H is a Hopf algebra generated by skew primitive elements. Namely, we show that if H is a character Hopf algebra (see [5]) over the field k of characteristic 0, and $\chi^h(g)$ is not an n^{th} primitive root of 1 ($n > 1$) for any character skew g-primitive element $h \in H$, then

3. every finite dimensional irreducible left $R\#H$-module is completely reducible as a left R-module;

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4. if R is left socular, then irreducible left $R\#H$-modules are completely reducible as left R-modules. Thus $\mathcal{J}(R) \subseteq \mathcal{J}(R\#H)$, where $\mathcal{J}$ is the Jacobson radical.

On the other hand, we should also point out that in the case where H is finite dimensional and pointed, there is a strong relationship between the Jacobson radicals of R and the crossed product $R\#H$; namely, it is proved in [7] that $\mathcal{J}(R\#H)^{\dim_k H} \subseteq \mathcal{J}(R) \cdot (R\#H)$.

We will now introduce the terminology and notation that will be used throughout the paper. Let R be an associative ring and σ be an automorphism of R. Then the additive map $\delta\colon R \to R$ is a σ-derivation if

$$\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$$

for all $a, b \in R$. Suppose that q is a nonzero central (σ, δ)-constant in R, i.e., $\sigma(q) = q$ and $\delta(q) = 0$. If $\delta\sigma = q\sigma\delta$, then δ is called a q-skew σ-derivation. If in addition R is a k-algebra, we assume that $q \in k^{\times}$. The following q-Leibniz Rules hold in R and $R[x; \sigma, \delta]$:

$$\delta(ab) = \sum_{i=0}^{n} \binom{n}{i}_q \sigma^{n-i}\delta^i(a)\delta^{n-i}(b) \text{ and } x^n a = \sum_{i=1}^{n} \binom{n}{i}_q \sigma^{n-i}\delta^i(a)x^{n-i}$$

for all $a, b \in R$ and $n \geqslant 0$. The Gaussian q-binomial coefficient $\binom{n}{i}_q$ is defined as the evaluation at $t = q$ of the polynomial function

$$\binom{n}{i}_t = \frac{(t^n - 1)(t^{n-1} - 1)\dots(t^{n-i+1} - 1)}{(t^i - 1)(t^{i-1} - 1)\dots(t - 1)}. \tag{1}$$

We will use the following q-Pascal identity:

$$\binom{n}{i}_q = \binom{n-1}{i}_q + q^{n-i}\binom{n-1}{i-1}_q = \binom{n-1}{i-1}_q + q^i\binom{n-1}{i}_q$$

for $n > i > 0$ (cf. [3]).

We will say that the ring R has q-characteristic zero if $1+q+\dots+q^m$ is invertible in R, for any integer $m \geqslant 1$. If in addition R is a k-algebra, then either q is not a root of unity or $q = 1$ and $\operatorname{char} k = 0$.

If $r \in R$, then a left R-module M is said to be *r-torsion free* if $rm \neq 0$ for all nonzero $m \in M$. If for any $m \in M$ there exists an integer $n = n(m)$ such that $r^n m = 0$, then M is called an *r-torsion* module.

A submodule E of an R-module M is said to be *essential* if $E \cap X \neq 0$ for any nonzero submodule $X \subseteq M$. It is well known that the intersection of all essential submodules of an R-module M is equal to the sum of all irreducible submodules of M and is called the socle of M, denoted by $\operatorname{Soc}(M)$. Finally, $\operatorname{Sing}(M)$ will be the singular submodule of M; that is, $\operatorname{Sing}(M) = \{m \in M \mid \operatorname{ann}_R(m) \text{ is essential in } {}_R R\}$.

2. m-SEQUENCES AND ESSENTIAL SUBMODULES

Let $R[x; \sigma, \delta]$ be a q-skew polynomial ring and M a left $R[x; \sigma, \delta]$-module. Let E be an essential R-submodule of M and $0 \neq m \in E$. By an *m-sequence* we mean a sequence $\mathbf{r} = \{r_n\}_{n \geqslant 0}$ of elements of R satisfying the following properties:

1° $\sigma^n(r_n)x^n m \in E$ for all $n \geqslant 0$ and $\sigma^s(r_s)x^s m \neq 0$ for some s;

2° if $\sigma^{n+1}(r_n)x^{n+1}m \in E$, then $r_{n+1} = r_n$;

3° if $\sigma^{n+1}(r_n)x^{n+1}m \notin E$, then $r_{n+1} \in Rr_n$ and $\sigma^{n+1}(r_{n+1})x^{n+1}m \in E \setminus \{0\}$.

The smallest integer s such that $\sigma^s(r_s)x^s m \neq 0$ we denote by $\deg \mathbf{r}$ and call the degree of $\mathbf{r}$.

Lemma 1. *If $a \in R$ and $\sigma^s(a)x^s m \neq 0$ for some $s \geqslant 1$, then there exists an m-sequence $\mathbf{r} = \{r_n\}_{n\geqslant 0}$ such that $r_0 = a$ and $\deg \mathbf{r} \leqslant s$.*

Proof. The sequence $\mathbf{r}$ we define inductively starting with $r_0 = \cdots = r_{i-1} = a$, where i is the smallest integer such that $\sigma^i(a)x^i m \notin E$. If such an i does not exist, the constant sequence $\mathbf{r} = \{a\}$ satisfies the desired property. Next suppose that $j \geqslant i-1$ and $r_0, \ldots, r_j$ are given. If $\sigma^{j+1}(r_j)x^{j+1}m \in E$, then we put $r_{j+1} = r_j$. If $\sigma^{j+1}(r_j)x^{j+1}m \notin E$, then by essentiality of E there exists $0 \neq c = \sigma^{j+1}(r') \in R$ such that

$$0 \neq c\sigma^{j+1}(r_j)x^{j+1}m = \sigma^{j+1}(r'r_j)x^{j+1}m \in E.$$

In this situation we put $r_{j+1} = r'r_j$. Clearly the sequence $\mathbf{r}$ satisfies conditions $1° - 3°$, and from the construction it follows immediately that $\deg \mathbf{r} \leqslant s$. □

An m-sequence $\mathbf{r} = \{r_n\}_{n\geqslant 0}$ is said to be **weak** if $r_j = r_{j+1}$ for some $j \geqslant \deg \mathbf{r}$. If $r_j \neq r_{j+1}$ for all $j \geqslant \deg \mathbf{r}$, we call $\mathbf{r}$ a **strict** m-sequence. Note that if $\mathbf{r}$ is strict and $j \geqslant \deg \mathbf{r}$, then $\sigma^j(r_j)x^j m \neq 0$. Indeed, if $\sigma^j(r_j)x^j m = 0$, then $\sigma^j(r_{j-1})x^j m$ must equal 0, and hence $r_j = r_{j-1}$.

Lemma 2. *Suppose that every m-sequence in R is strict. Then:*

(1) *if $a \in R$ is such that $0 \neq ax^l m \in E$, then $\sigma(a)x^{l+1}m \notin E$;*

(2) *if $\mathbf{r} = \{r_n\}_{n\geqslant 0}$ is an m-sequence and $l \geqslant \deg \mathbf{r}$, then $\sigma^j(r_l)x^j m = 0$ for all $j < l$;*

(3) *$\operatorname{ann}(x^{j+1}m) \subseteq \sigma^{-1}(\operatorname{ann}(x^j m))$ for all $j \geqslant 0$.*

Proof. 1. Suppose that $0 \neq ax^l m \in E$ and $\sigma(a)x^{l+1}m \in E$. By Lemma 1 we can take an m-sequence $\mathbf{r}$ such that $r_0 = \sigma^{-l}(a)$ and $\deg \mathbf{r} \leqslant l$. Then $r_l = br_0 = b\sigma^{-l}(a)$, where $b \in R$. Notice that

$$\sigma^{l+1}(r_l)x^{l+1}m = \sigma^{l+1}(b)\sigma(a)x^{l+1}m \in E.$$

Hence $r_l = r_{l+1}$, contradicting our assumption that every m-sequence in R is strict.

2. Suppose that $\sigma^j(r_l)x^j m \neq 0$ for some $j < l$. From the definition of an m-sequence it follows that we can choose $a, b \in R$ such that $r_l = ar_j = br_{j+1}$. Then $0 \neq \sigma^j(r_l)x^j m = \sigma^j(a)\sigma^j(r_j)x^j m \in E$. On the other hand, $\sigma^{j+1}(r_l)x^{j+1}m = \sigma^{j+1}(b)\sigma^{j+1}(r_{j+1})x^{j+1}m \in E$, which is impossible by 1.

3. Suppose $a \in R$ is such that $\sigma(a)x^{j+1}m = 0$. By item 1, it follows that either $ax^j m = 0$ or $ax^j m \notin E$. If $ax^j m \notin E$, then there exists $r \in R$ such $0 \neq rax^j m \in E$. But in this situation $0 = \sigma(ra)x^{j+1}m \in E$. By item 1 we obtain that $ax^j m$ must be equal to 0; thus $\operatorname{ann}(x^{j+1}m) \subseteq \sigma^{-1}(\operatorname{ann}(x^j m))$. □

Corollary 3. *If every m-sequence in R is strict, then R contains an infinite strictly descending chain of left ideals*

$$\operatorname{ann}(m) \supsetneq \sigma^{-1}(\operatorname{ann}(xm)) \supsetneq \cdots \supsetneq \sigma^{-l}(\operatorname{ann}(x^l m)) \supsetneq \cdots.$$

Proof. Lemma 2(3) implies that $\sigma^{-l}(\operatorname{ann}(x^l m)) \subseteq \sigma^{-(l-1)}(\operatorname{ann}(x^{l-1}m))$ for any $l > 0$. To see that the inclusion is strict, it is enough to consider an m-sequence $\mathbf{r}$ of degree $\leqslant l-1$. Then Lemma 2(2) yields that $r_l \in \sigma^{-(l-1)}(\operatorname{ann}(x^{l-1}m))$, but clearly $r_l \notin \sigma^{-l}(\operatorname{ann}(x^l m))$. □

Lemma 4. *If R contains a weak m-sequence, then there exists an element $r \in R$ and a nonnegative integer n such that*

1. $0 \neq \sigma^n(r)x^n m \in E$ *and* $\sigma^{n+1}(r)x^{n+1}m \in E$,
2. $rm = \sigma(r)xm = \cdots = \sigma^{n-1}(r)x^{n-1} = 0$.

Proof. Let $l \geqslant \deg(\mathbf{r})$ be the smallest integer with respect to the equality $r_l = r_{l+1}$. Then $\sigma^l(r_l)x^l m \neq 0$. Otherwise, if $\sigma^l(r_l)x^l m = 0$, then from the definition it follows that $\sigma^l(r_{l-1})x^l m \in E$, and hence $r_{l-1} = r_l$. Next consider the smallest integer n with respect to $\sigma^n(r_l)x^n m \neq 0$. It is clear that $n \leqslant l$. Note that if $j \leqslant l$, then $r_l = s_j r_j$ for some $s_j \in R$. Thus $\sigma^j(r_l)x^j m = \sigma^j(s_j)\sigma^j(r_j)x^j m \in E$. Therefore $r = r_l$ and n satisfy the lemma. □

Lemma 5. *Let M be a q-torsion free left $R[x;\sigma,\delta]$-module and $r \in R$, $m \in M$ be such that*

$$rm = \sigma(r)xm = \cdots = \sigma^{n-1}(r)x^{n-1}m = 0.$$

Then $\sigma^i\delta^j(r)x^i m = 0$ if $i + j \leqslant n - 1$, and $\sigma^n(r)x^n m = (-1)^n q^{\frac{n(n-1)}{2}}\delta^n(r)m$.

Proof. First we show that if i, j are nonnegative integers and $i + j \leqslant n - 1$, then $\sigma^i\delta^j(r)x^i m = 0$.

Suppose that $\sigma^i\delta^j(r)x^i m \neq 0$ and take i, j such that the sum $i + j$ is possibly minimal. Next take j possibly minimal. By assumption it follows that $j > 0$, so

$$\sigma^{i+1}\delta^{j-1}(r)x^{i+1}m = 0 \quad \text{and} \quad \sigma^i\delta^{j-1}(r)x^i m = 0.$$

Thus

$$\begin{aligned} 0 = x(\sigma^i\delta^{j-1}(r)x^i m) &= \sigma^{i+1}\delta^{j-1}(r)x^{i+1}m + \delta\sigma^i\delta^{j-1}(r)x^i m \\ &= q^i\sigma^i\delta^j(r)x^i m, \end{aligned}$$

a contradiction. The above implies, in particular, that if $i + j = n - 1$, then

$$0 = x(\sigma^i\delta^j(r)x^i m) = \sigma^{i+1}\delta^j(r)x^{i+1}m + q^i\sigma^i\delta^{j+1}(r)x^i m.$$

Hence

$$\begin{aligned} \sigma^n(r)x^n m = -q^{n-1}\sigma^{n-1}\delta(r)x^{n-1}m &= q^{n-1}q^{n-2}\sigma^{n-2}\delta^2(r)x^{n-2}m \\ = \cdots = (-1)^n q^{n-1}q^{n-2}\ldots q\delta^n(r)m &= (-1)^n q^{\frac{n(n-1)}{2}}\delta^n(r)m. \end{aligned}$$

□

For $1 \leqslant i, j \leqslant n$ let $a_{ij} = \binom{i+1}{j}_q q^{(n-i)j}$, where $\binom{i+1}{j}_q$ denotes the Gaussian q-binomial coefficient (see (1) in the Introduction). Let

$$D_n = \det[a_{ij}] = \det \begin{bmatrix} \binom{2}{1}_q q^{n-1} & \binom{2}{2}_q q^{2(n-1)} & 0 & \ldots & 0 \\ \binom{3}{1}_q q^{n-2} & \binom{3}{2}_q q^{2(n-2)} & \binom{3}{3}_q q^{3(n-2)} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \binom{n}{1}_q q & \binom{n}{2}_q q^2 & \binom{n}{3}_q q^3 & \ldots & \binom{n}{n}_q q^n \\ \binom{n+1}{1}_q & \binom{n+1}{2}_q & \binom{n+1}{3}_q & \ldots & \binom{n+1}{n}_q \end{bmatrix}.$$

Lemma 6. $D_n = q^{\frac{n^3-n}{6}}(1 + q + \cdots + q^n)$.

Proof. Notice that by using the q-Pascal identity,

$$a_{i+1,j} = \binom{i+2}{j}_q q^{(n-i-1)j} = \binom{i+1}{j-1}_q q^{(n-i-1)j} + \binom{i+1}{j}_q q^j q^{(n-i-1)j}$$
$$= \binom{i+1}{j-1}_q q^{(n-i-1)j} + a_{ij}.$$

The above implies that

$$D_n = \det \begin{bmatrix} \binom{2}{1}_q q^{n-1} & \binom{2}{2}_q q^{2(n-1)} & 0 & \dots & 0 \\ \binom{2}{0}_q q^{n-2} & \binom{2}{1}_q q^{2(n-2)} & \binom{2}{2}_q q^{3(n-2)} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \binom{n-1}{0}_q q & \binom{n-1}{1}_q q^2 & \binom{n-1}{2}_q q^3 & \dots & \binom{n-1}{n-1}_q q^n \\ \binom{n}{0}_q & \binom{n}{1}_q & \binom{n}{2}_q & \dots & \binom{n}{n-1}_q \end{bmatrix}$$

$$= \binom{2}{1}_q q^{n-1} q^{n-2} \dots q \cdot D_{n-1} - \binom{2}{2}_q q^{2(n-1)} W_{n-1},$$

where

$$W_{n-1} = \det \begin{bmatrix} q^{n-2} & \binom{2}{2}_q q^{3(n-2)} & 0 & \dots & 0 \\ q^{n-3} & \binom{3}{2}_q q^{3(n-3)} & \binom{3}{3}_q q^{4(n-3)} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ q & \binom{n-1}{2}_q q^3 & \binom{n-1}{3}_q q^4 & \dots & \binom{n-1}{n-1}_q q^n \\ 1 & \binom{n}{2}_q & \binom{n}{3}_q & \dots & \binom{n}{n-1}_q \end{bmatrix}.$$

Again applying the q-Pascal identity, one immediately obtains that

$$W_{n-1} = \det \begin{bmatrix} q^{n-2} & \binom{2}{2}_q q^{3(n-2)} & 0 & \dots & 0 \\ 0 & \binom{2}{1}_q q^{3(n-3)} & \binom{2}{2}_q q^{4(n-3)} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \binom{n-2}{1}_q q^3 & \binom{n-2}{2}_q q^4 & \dots & \binom{n-2}{n-2}_q q^n \\ 0 & \binom{n-1}{1}_q & \binom{n-1}{2}_q & \dots & \binom{n-1}{n-2}_q \end{bmatrix}$$

$$= q^{n-2} q^{2(n-3)} q^{2(n-4)} \dots q^2 \cdot D_{n-2} = q^{n^2-4n+4} D_{n-2}.$$

Thus

$$D_n = (1+q) q^{\frac{n(n-1)}{2}} D_{n-1} - q^{n^2-2n+2} D_{n-2}$$

with $D_1 = 1+q$ and $D_2 = q(1+q+q^2)$. The lemma now follows by an easy induction. □

Proposition 7. *Let M be a left $R[x;\sigma,\delta]$-module which is D_n-torsion free for all $n \geqslant 1$. Let E be an essential R-submodule of M such that for every $m \in E$, the ring R contains a weak m-sequence. Then*

$$E \cap x^{-1}E = \{m \in E \mid xm \in E\}$$

is also essential as an R-submodule.

Proof. Notice that if $e \in E$ and $xe \in E$, then for every $r \in R$,

$$xre = \sigma(r)xe + \delta(r)e \in E.$$

Thus $E \cap x^{-1}E$ is an R-submodule of M.

Suppose that $E \cap x^{-1}E$ is not essential. Then there exists a nonzero element $m \in E$ such that $(E \cap x^{-1}E) \cap Rm = 0$. Since R contains a weak m-sequence, by Lemma 4 we can take $r \in R$ and $n \geqslant 0$ such that

$$rm = \sigma(r)xm = \cdots = \sigma^{n-1}(r)x^{n-1}m = 0,$$

$$0 \neq \sigma^n(r)x^n m \in E \quad \text{and} \quad \sigma^{n+1}(r)x^{n+1}m \in E.$$

For $1 \leqslant i, j \leqslant n$, let $a_{ij} = \binom{i+1}{j}_q q^{(n-i)j}$ and $x_j = \sigma^{n+1-j}\delta^j(r)x^{n+1-j}m$. Applying the q-Leibniz rule for $i = 1, 2, \ldots, n-1$, we obtain

$$\begin{aligned} 0 = x^{i+1}(\sigma^{n-i}(r)x^{n-i}m) &= \sum_{j=0}^{i+1} \binom{i+1}{j}_q \sigma^{i+1-j}\delta^j\sigma^{n-i}(r)x^{n+1-j}m \\ &= \sum_{j=0}^{i+1} \binom{i+1}{j}_q q^{(n-i)j}\sigma^{n+1-j}\delta^j(r)x^{n+1-j}m \\ &= \sigma^{n+1}(r)x^{n+1}m + \sum_{j=1}^{i+1} a_{ij}x_j. \end{aligned}$$

Thus $\sum\limits_{j=1}^{i+1} a_{ij}x_j = -\sigma^{n+1}(r)x^{n+1}m \in E$. Moreover, for $i = n$ we have

$$0 = x^{n+1}rm = \sigma^{n+1}(r)x^{n+1}m + \sum_{j=1}^{n} a_{nj}x_j + \delta^{n+1}(r)m,$$

so $\sum\limits_{j=1}^{n} a_{nj}x_j \in E$. Now it is clear that for any $j = 1, 2, \ldots, n$ the element $D_n x_j \in E$, where D_n is the determinant from Lemma 6. We note that $D_n x_1 = D_n\sigma^n\delta(r)x^n m \in E$, so

$$\begin{aligned} x(D_n\sigma^n(r)x^n m) &= D_n\sigma^{n+1}(r)x^{n+1}m + D_n\delta\sigma^n(r)x^n m \\ &= D_n\sigma^{n+1}(r)x^{n+1}m + D_n q^n\sigma^n\delta(r)x^n m \in E. \end{aligned}$$

On the other hand, by Lemma 5, $\sigma^n(r)x^n m = (-1)^n q^{\frac{n(n-1)}{2}}\delta^n(r)m$ and M is D_n-torsion free; thus

$$0 \neq D_n\sigma^n(r)x^n m \in (E \cap x^{-1}E) \cap Rm,$$

a contradiction. Therefore $E \cap x^{-1}E$ is an essential submodule of M. □

Corollary 8. *Let M be a left $R[x;\sigma,\delta]$-module which is D_n-torsion free for all $n \geqslant 1$. Suppose that for every essential R-submodule E of M and $0 \neq m \in E$, the ring R contains a weak m-sequence. Then* $\mathrm{Soc}({}_RM)$ *is an $R[x;\sigma,\delta]$-module. In particular, if M is simple as an $R[x;\sigma,\delta]$-module, then either* $\mathrm{Soc}({}_RM) = 0$ *or ${}_RM$ is completely reducible.*

Proof. Let $m \in \mathrm{Soc}({}_RM)$. If E is an essential submodule of ${}_RM$, then by Proposition 7 $E \cap x^{-1}E$ is also essential, so $m \in E \cap x^{-1}E$. Hence $xm \in E$. Therefore $\mathrm{Soc}({}_RM)$ is an $R[x;\sigma,\delta]$-module. □

3. Applications

In this section we describe situations in which our condition on the existence of weak m-sequences is automatically satisfied.

Let Λ be a well ordered set of ordinal numbers with the least element 0. For a ring R one can define a chain of ideals $\{S_\alpha\}_{\alpha\in\Lambda}$ as follows: $S_0 = 0$; if $\alpha \in \Lambda$, then $S_{\alpha+1}/S_\alpha = \mathrm{Soc}(R/S_\alpha)$, the left socle of R/S_α. If $\beta \in \Lambda$ is a limit number, set $S_\beta = \bigcup_{\alpha<\beta} S_\alpha$. Recall that a ring R is said to be left *socular* (cf. [1]) if every nonzero left R-module contains a simple submodule. If R is left socular, the set Λ can be chosen such that $R = S_\alpha$ for some $\alpha \in \Lambda$. Note that the class of socular rings contains left artinian rings and right perfect rings.

If A is a k-algebra, then A-module M is *locally finite dimensional* if every finitely generated submodule of M is finite dimensional.

Proposition 9. *Let M be a left $R[x;\sigma,\delta]$-module and E its essential R-submodule. Suppose that one of the following conditions is fulfilled:*

1. *R is left socular;*
2. *R is a left noetherian k-algebra and M is locally finite dimensional as a $k[x]$-module;*
3. $\dim_k M < \infty$*;*
4. *there exists an integer N such that $d^{N+1}(r) \in \sum_{j=0}^{N} Rd^j(r)$ for all $r \in R$;*
5. *M is x-torsion; i.e., for any $m \in M$ there exists $n = n(m)$ such that $x^n m = 0$;*
6. *R is a k-algebra, $\sigma = \mathrm{id}_R$ and M is locally finite dimensional as a $k[x]$-module.*

Then for any nonzero $m \in E$ the ring R contains a weak m-sequence.

Proof. **1.** Suppose that R is left socular. Let γ be the smallest ordinal such that S_γ contains an m-sequence $\{r_l\}_{l\geqslant 0}$. It is clear that γ is not a limit ordinal. Note that if $a \in S_{\gamma-1}$, then $\sigma^l(a)x^l m = 0$. Otherwise, we have an m-sequence $\{r'_l\}_{l\geqslant 0}$ with $r'_0 = a \in S_{\gamma-1}$. Since $Rr'_l \supseteq Rr'_{l+1}$, one obtains that $r'_l \in S_{\gamma-1}$ for all l. This contradicts minimality of γ.

Let $\varphi\colon R \to R/S_{\gamma-1}$ be the canonical homomorphism. Since $Rr_0 \supseteq Rr_1 \supseteq \cdots \supseteq Rr_l \supseteq \cdots$, we have a chain

$$\varphi(Rr_0) \supseteq \varphi(Rr_1) \supseteq \cdots \supseteq \varphi(Rr_l) \supseteq \cdots$$

of cyclic submodules of a semisimple module $S_\gamma/S_{\gamma-1}$. Since $\varphi(Rr_0)$ is contained in a finite direct sum of simple modules, this chain terminates. On the other hand, if $\varphi(Rr_l) = \varphi(Rr_{l+1})$, then there exist $r' \in R$ and $a \in S_{\gamma-1}$ such that $r_l = r'r_{l+1} + a$. By the above, $\sigma^{l+1}(a)x^{l+1}m = 0$, so

$$\sigma^{l+1}(r_l)x^{l+1}m = \sigma^{l+1}(r')\sigma^{l+1}(r_{l+1})x^{l+1}m \in E.$$

From the definition of an m-sequence it follows that $r_l = r_{l+1}$. Therefore the sequence $\mathbf{r}$ is weak.

2. Suppose that every m-sequence in R is strict. Corollary 3 tells us that the chain of left ideals

$$\mathrm{ann}(m) \supsetneqq \sigma^{-1}(\mathrm{ann}(xm)) \supsetneqq \cdots \supsetneqq \sigma^{-l}(\mathrm{ann}(x^l m)) \supsetneqq \cdots$$

is strict. Since $\dim \operatorname{span}_F(m, xm, x^2m, \dots) < \infty$, there exists an integer t such that $x^n m \in \operatorname{span}_F(m, xm, x^2m, \dots, x^t m)$ for all $n \geqslant t$. Then

$$\operatorname{ann}(m, xm, x^2m, \dots, x^t m) \subseteq \operatorname{ann}(x^n m)$$

for $n \geqslant t$, and consequently $\bigcap_{l=0}^{\infty} \operatorname{ann}(x^l m) = \bigcap_{l=0}^{t} \operatorname{ann}(x^l m)$. Set $I = \bigcap_{l=0}^{t} \operatorname{ann}(x^l m)$ and take $r \in I$. For any $l \geqslant 1$, $r \in \operatorname{ann}(x^l m)$, so

$$\sigma^{-l}(r) \in \sigma^{-l}(\operatorname{ann}(x^l m)) \subseteq \sigma^{-(l-1)}(\operatorname{ann}(x^{l-1}m));$$

hence $\sigma^{-1}(r) \in \operatorname{ann}(x^{l-1}m)$. Then it follows that $\sigma^{-1}(I) \subseteq I$, and so $I \subseteq \sigma(I)$. The ring R is left noetherian, so the chain $I \subseteq \sigma(I) \subseteq \sigma^2(I) \dots$ must stop. It implies immediately that $\sigma(I) = I$.

Next we claim that there exists an increasing sequence $\{f(n)\}_{n \geqslant 0}$ of nonnegative integers such that

$$\sigma\left(\bigcap_{l=0}^{f(n)} \operatorname{ann}(x^l m)\right) \nsubseteq \bigcap_{j > f(n)} \operatorname{ann}(x^j m).$$

We proceed by induction. By Corollary 3 we can put $f(0) = 0$. Assume $n \geqslant 0$ and let $a \in \bigcap_{l=0}^{f(n)} \operatorname{ann}(x^l m)$ be such that $\sigma(a)x^i m \neq 0$ for some $i > f(n)$. Since I is σ-stable, $a \notin I$, so there exists $s > f(n)$ such that $a \in \bigcap_{l=0}^{s-1} \operatorname{ann}(x^l m)$ and $ax^s m \neq 0$. Take $b \in R$ such that $0 \neq bax^s m \in E$. If every m-sequence is strict, then by Lemma 2(1), $\sigma(ba)x^{s+1}m \notin E$. Since E is essential, one can choose $c \in R$ such that $0 \neq \sigma(cba)x^{s+1}m \in E$. Again by Lemma 2(1), $cbax^s m = 0$, so $cba \in \bigcap_{l=0}^{s} \operatorname{ann}(x^l m)$. Since $\sigma(cba)x^{s+1}m \neq 0$, we have $\sigma\left(\bigcap_{l=0}^{s} \operatorname{ann}(x^l m)\right) \nsubseteq \bigcap_{j>s} \operatorname{ann}(x^j m)$. Thus it suffices to put $f(n+1) = s$. This proves the claim.

But now, if $f(n) > t$, then $I = \bigcap_{l=0}^{f(n)} \operatorname{ann}(x^l m) = \bigcap_{l=0}^{\infty} \operatorname{ann}(x^l m)$. Since I is σ-stable,

$$\sigma\left(\bigcap_{l=0}^{f(n)} \operatorname{ann}(x^l m)\right) \subseteq \bigcap_{l=0}^{\infty} \operatorname{ann}(x^l m) \subseteq \bigcap_{j > f(n)} \operatorname{ann}(x^j m),$$

contradicting the definition of $f(n)$. Thus R contains a weak m-sequence.

3. Let $P = \operatorname{ann}(M)$. Then $\dim_F(R/P) < \infty$ and $P \subseteq \operatorname{ann}(x^l m)$ for any l. Note that the mapping $a + \operatorname{ann}(x^l m) \longmapsto \sigma^{-l}(a) + \sigma^{-l}(\operatorname{ann}(x^l m))$ provides an isomorphism of vector spaces $R/\operatorname{ann}(x^l m) \approx R/\sigma^{-l}(\operatorname{ann}(x^l m))$. Thus

$$\dim_F R/\sigma^{-l}(\operatorname{ann}(x^l m)) \leqslant \dim_F(R/P).$$

From Corollary 3 it follows that R contains a weak m-sequence.

4. Let $\mathbf{r} = \{r_n\}_{n \geqslant 0}$ be a strict m-sequence with $\deg \mathbf{r} \leqslant N$. Then $\sigma^j(r_{N+1})x^j m = 0$ for all $j \leqslant N$ and $\sigma^{N+1}(r_{N+1})x^{N+1}m \neq 0$. By Lemma 5,

$$0 = \sigma^j(r_{N+1})x^j m = (-1)^j q^{\frac{j(j-1)}{2}} \delta^j(r)m$$

for all $j \leqslant N$. Thus

$$\sigma^{N+1}(r_{N+1})x^{N+1}m = (-1)^{N+1}\frac{N(N+1)}{2}\delta^{N+1}(r_{N+1})m$$
$$\in \sum_{j=0}^{N} R\delta^j(r_{N+1})m = 0,$$

a contradiction. Consequently, in this situation, every m-sequence is weak.

5. This follows directly from Corollary 3.

6. Suppose $\sigma = \mathrm{id}_R$. If every m-sequence in R is strict, Corollary 3 says that the chain $\mathrm{ann}(m) \supsetneq \mathrm{ann}(xm) \supsetneq \cdots \supsetneq \mathrm{ann}(x^n m) \supsetneq \cdots$ is strict. But this contradicts our assumption that $\mathrm{span}_F\{m, xm, \dots, x^l m \dots\}$ is finite dimensional. □

Recall that an automorphism σ of the ring R is said to be of locally finite order if for every $r \in R$, there exists an integer $n = n(r) > 0$ such that $\sigma^n(r) = r$. If the ring R is left socular, then nonzero left R-modules contain simple submodules. Therefore Proposition 9, condition 1, and Corollary 8 give us

Corollary 10. *If R is a left socular ring of q-characteristic zero, then simple left $R[x;\sigma,\delta]$-modules are completely reducible as left R-modules. Thus the Jacobson radical $\mathcal{J}(R)$ is contained in the Jacobson radical $\mathcal{J}(R[x;\sigma,\delta])$. Moreover, if the automorphism σ has locally finite order, then*

$$\mathcal{J}(R[x;\sigma,\delta]) = \mathcal{J}(R)[x;\sigma,\delta].$$

Proof. Since simple $R[x;\sigma,\delta]$-modules are completely reducible as R-modules, we have $\mathcal{J}(R) \subseteq \mathcal{J}(R[x;\sigma,\delta])$. Suppose that σ has locally finite order. We know that $\mathcal{J}(R[x;\sigma,\delta]) \cap R$ is a quasi-regular ideal of R, so $\mathcal{J}(R[x;\sigma,\delta]) \cap R \subseteq \mathcal{J}(R)$ and consequently $\mathcal{J}(R[x;\sigma,\delta]) \cap R = \mathcal{J}(R)$. This implies that $\mathcal{J}(R)$ is δ-stable and

$$R[x;\sigma,\delta]/\mathcal{J}(R)[x;\sigma,\delta] \simeq (R/\mathcal{J}(R))[x;\widehat{\sigma},\widehat{\delta}],$$

where $\widehat{\sigma}$ is an induced automorphism and $\widehat{\delta}$ is a q-skew $\widehat{\sigma}$-derivation of $R/\mathcal{J}(R)$, respectively. Now it remains to prove that if R is semiprimitive and socular, then $S = R[x;\sigma,\delta]$ is semiprimitive. To this end, suppose that $\mathcal{J}(S) \neq 0$ and let n be the minimum of degrees of nonzero polynomials from $\mathcal{J}(S)$. The set $\{0\} \cup \{a \mid ax^n + g(x) \in \mathcal{J}(S), \text{where } \deg g(x) < n\}$ is a nonzero ideal of R. In particular, it contains a minimal left ideal of the form $I = Re$, where e is a nonzero idempotent. Let $f(x) = ex^n + g(x) \in \mathcal{J}(S)$ and $m > 0$ be such that $\sigma^m(e) = e$. By eventually replacing $f(x)$ by $f(x)x^k$, where k is such that $\deg f(x)x^k$ is divisible by m, we have in the Jacobson radical of S a nonzero polynomial $f(x) = ex^l + h(x)$ such that e is a nonzero idempotent, $\sigma^l(e) = e$, and $\deg h(x) < l$. It is well known that $\mathcal{J}(eSe) = e\mathcal{J}(S)e$. Therefore

$$ef(x)e = ex^l e + eh(x)e = ex^l + \widetilde{h}(x) \in \mathcal{J}(eSe),$$

where $\widetilde{h}(x) \in eSe$. Let $eg(x)e \in eSe$ be a quasi-inverse for $ef(x)e$. Then $eg(x)e$ has a positive degree s in x and

$$ef(x)e + eg(x)e = ef(x)eg(x)e.$$

Since e is the identity element in eSe, the right-hand side of the above equality has degree $n + s > \max\{n, s\} \geq \deg(ef(x)e + eg(x)e)$. Thus $\mathcal{J}(S) = 0$. □

In [6] the authors considered the so-called "finite Jacobson radical" $\mathcal{J}_{fin}(R)$ of a k-algebra R, defined as the intersection of all the annihilators of all finite dimensional irreducible (left) R-modules. Thus by Proposition 9, condition 3, and Corollary 8 we have

Corollary 11. *Let R be a k-algebra with a q-skew σ-derivation δ. If R has q-characteristic zero, then every finite dimensional irreducible left $R[x;\sigma,\delta]$-module is completely reducible as a left R-module. Thus*

$$\mathcal{J}_{fin}(R) \subseteq \mathcal{J}_{fin}(R[x;\sigma,\delta]).$$

We note that R can be viewed as a left $R[x;\sigma,\delta]$-module with the action defined as

$$(\sum_i a_i x^i).r = \sum_i a_i \delta^i(r).$$

The $R[x;\sigma,\delta]$-submodules of R are precisely the left ideals of R which are stable under δ. Recall that δ is said to be locally algebraic if R is locally finite dimensional as a left $k[x]$-module. Moreover in this case, if $m \in R$, then $\sigma^{-l}(\mathrm{ann}_R(x^l m)) = \mathrm{ann}_R(\delta^l(\sigma^{-l}(m))$. Thus if R satisfies descending chain condition on left annihilators, then Corollary 3 guarantees that for any essential left ideal E and a nonzero element $m \in E$, the ring R contains a weak m-sequence. Therefore we can apply Propositions 7, 9 and Corollary 8 to obtain the following.

Corollary 12. *Let R be a k-algebra of q-characteristic zero, with a q-skew σ-derivation δ. Suppose that one of the following conditions is fulfilled:*

1. *R satisfies dcc on left annihilators;*
2. *R is left noetherian and δ is locally algebraic;*
3. *δ is locally nilpotent;*
4. *there exists an integer N such that for any $r \in R$, $\delta^{N+1}(r) \in \sum_{j=0}^{N} R\delta^j(r)$;*
5. *$\sigma = \mathrm{id}_R$, $q = 1$ and the derivation δ is locally algebraic.*

If M is a left $R[x;\sigma,\delta]$-module, then the singular submodule $\mathrm{Sing}({}_RM)$ over R is also an $R[x;\sigma,\delta]$-submodule. The left socle $\mathrm{Soc}({}_RR)$ of R and left singular ideal $\mathrm{Sing}({}_RR)$ are δ-invariant. In addition, if R contains a minimal left ideal and R does not contain proper δ-stable two-sided ideals, then R is a semisimple artinian ring.

Proof. Let $m \in \mathrm{Sing}({}_RM)$ and $L = \mathrm{ann}_R(m)$. If L is an essential left ideal of R, then by Proposition 7, $\widehat{L} = L \cap \delta^{-1}(L) = \{r \in L \mid \delta(r) \in L\}$ is essential. It is also clear that $\sigma(\widehat{L})$ is essential, and for every $r \in \widehat{L}$,

$$\sigma(r)xm = xrm - \delta(r)m = 0.$$

Hence $\sigma(\widehat{L}) \subseteq \mathrm{ann}_R(xm)$ and $xm \in \mathrm{Sing}({}_RM)$. Consequently, $\mathrm{Sing}({}_RM)$ is an $R[x;\sigma,\delta]$-submodule of M.

If R contains a minimal ideal, then $\mathrm{Soc}({}_RR)$ is a nonzero and δ-stable ideal of R. Therefore if R is δ-simple, then $R = \mathrm{Soc}({}_RR)$. Since R has unity, R is a finite direct sum of minimal left ideals. □

Let H be a Hopf algebra with comultiplication Δ and with the group G of group-like elements, i.e., $G = \{g \in H \mid \Delta(g) = g \otimes g\}$. For $g \in G$, let

$$L_g = \{h \in H \mid \Delta(h) = h \otimes 1 + g \otimes h\}$$

be the subspace of g-primitive (skew primitive) elements. It is clear that the group G acts on H by the conjugations $h^g = g^{-1}hg$ and that the subspace $L = \bigoplus_{g\in G} L_g$ is G-stable under this action. Following [5], recall that an element $h \in H$ is said to be a *character element* if there exists a character $\chi\colon G \to k^\times$ such that for all $g \in G$,

$$g^{-1}hg = \chi(g)h.$$

If h is a nonzero character element, then the character χ is uniquely determined by the above equality, and $\chi = \chi^h$ is called a *weight* of h. A Hopf algebra H is called a *character* if the group G is abelian and H is generated as an algebra with unity by character skew primitive elements. This is a large class of Hopf algebras containing, among others, quantum planes, Drinfeld-Jimbo quantized enveloping algebras $U_q(\mathfrak{g})$, and G-universal enveloping algebras of Lie color algebras.

If R is an associative algebra acted on by a character Hopf algebra H, then any character skew primitive element $h \in L_g$ acts on R as a $\chi^h(g)$-skew g-derivation. In this situation, any left module M over the smash product $R\#H$ is a module over the skew polynomial ring $R[x; g, h]$, where the action of x coincides with the action of h, i.e., $x.m = hm$. Therefore, we are in a position to apply Propositions 7, 9 and Corollary 8 to actions of character Hopf algebras.

Theorem 13. *Let H be a character Hopf algebra over the field k of characteristic 0 and suppose that $\chi^h(g)$ is not an n^{th} primitive root of unity $(n > 1)$ for any character skew primitive element $h \in L_g$ and $g \in G$. Let R be an associative H-module algebra. Then:*

(1) *Every finite dimensional irreducible left $R\#H$-module is completely reducible as a left R-module. In particular, $\mathcal{J}_{fin}(R) \subseteq \mathcal{J}_{fin}(R\#H)$.*
(2) *If R is left socular, then irreducible left $R\#H$-modules are completely reducible as left R-modules. Thus $\mathcal{J}(R) \subseteq \mathcal{J}(R\#H)$.*

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